

ALGEBRAIC DEPENDENCES AND UNIQUENESS PROBLEM OF MEROMORPHIC MAPPINGS SHARING MOVING HYPERPLANES WITHOUT COUNTING MULTIPLICITIES

LE NGOC QUYNH

ABSTRACT. This article deals with the multiple values and algebraic dependences problem of meromorphic mappings sharing moving hyperplanes in projective space. We give some algebraic dependences theorems for meromorphic mappings sharing moving hyperplanes without counting multiplicity, where all zeros with multiplicities more than a certain number are omitted. Basing on these results, some unicity theorems regardless of multiplicity for meromorphic mappings in several complex variables are given. These results are extensions and strong improvements of some recent results.

1. INTRODUCTION

The theory on algebraic dependences of meromorphic mappings in several complex variables into the complex projective spaces for fixed targets was first studied by W. Stoll [7]. Later, M. Ru [6] generalized W. Stoll's result to the case of holomorphic curves into the complex projective spaces sharing moving hyperplanes. Recently, by using the new second main theorem given by Thai-Quang [11], P. D. Thoan, P. V. Duc and S. D. Quang [3, 10, 8] gave some improvements of the results of W. Stoll and M. Ru. In order to state some of their result, we first recall the following.

We call a meromorphic mapping of \mathbf{C}^n into $\mathbf{P}^N(\mathbf{C})^*$ a moving hyperplane in $\mathbf{P}^N(\mathbf{C})$. Let a_1, \dots, a_q ($q \geq N + 1$) be q moving hyperplanes with reduced representations $a_j = (a_{j0} : \dots : a_{jN})$ ($1 \leq j \leq q$). We say that a_1, \dots, a_q are in general position if $\det(a_{j_k l}) \neq 0$ for any $1 \leq j_0 < j_1 < \dots < j_N \leq q$.

Let $f_i : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ ($1 \leq i \leq \lambda$) be meromorphic mappings with reduced representations $f_i := (f_{i0} : \dots : f_{iN})$. Let $g_j : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})^*$ ($0 \leq j \leq q - 1$) be meromorphic mappings in general position with reduced representations $g_j := (g_{j0} : \dots : g_{jN})$. Put $(f_i, g_j) := \sum_{s=0}^N f_{is} g_{js} \neq 0$ for each $1 \leq i \leq \lambda$, $0 \leq j \leq q - 1$ and assume that $\min\{1, \nu_{(f_1, g_j), \leq k_j}^0\} = \dots = \min\{1, \nu_{(f_\lambda, g_j), \leq k_j}^0\}$. Put $A_j = \text{Supp}(\nu_{(f_1, g_j), \leq k_j}^0)$. Assume that each A_j has an irreducible decomposition as follows $A_j = \bigcup_{s=1}^{t_j} A_{js}$. Set $A = \bigcup_{A_{js} \neq A_{j's'}} \{A_{js} \cap A_{j's'}\}$ with $1 \leq s \leq t_j$, $1 \leq s' \leq t_{j'}$, $0 \leq j, j' \leq q - 1$.

Denote by $T[N + 1, q]$ the set of all injective maps from $\{1, \dots, N + 1\}$ to $\{0, \dots, q - 1\}$. For each $z \in \mathbf{C}^n \setminus \{\bigcup_{\beta \in T[N+1, q]} \{z | g_{\beta(1)}(z) \wedge \dots \wedge g_{\beta(N+1)}(z) = 0\} \cup A \cup \bigcup_{i=1}^\lambda I(f_i)\}$, we define $\rho(z) = \#\{j | z \in A_j\}$. Then $\rho(z) \leq N$. For any positive number $r > 0$, define $\rho(r) =$

2010 Mathematics Subject Classification: Primary 32H30, 32A22; Secondary 30D35.

Key words and phrases: algebraic dependence, unicity problem, meromorphic mapping, truncated multiplicity.

$\sup\{\rho(z) \mid |z| \leq r\}$, where the supremum is taken over all $z \in \mathbf{C}^n \setminus \{\bigcup_{\beta \in T[N+1, q]} \{z \mid g_{\beta(1)}(z) \wedge \cdots \wedge g_{\beta(N+1)}(z) = 0\} \cup A \cup \bigcup_{i=1}^{\lambda} I(f_i)\}$. Then $\rho(r)$ is a decreasing function. Let

$$d := \lim_{r \rightarrow +\infty} \rho(r).$$

Then $d \leq N$. If for each $i \neq j$, $\dim\{A_i \cap A_j\} \leq n - 2$, then $d = 1$.

When all $k_j = +\infty$, in [3], P. V. Duc and P. D. Thoan proved the following.

Theorem A (see [3, Theorem 1]). *Let $f_1, \dots, f_{\lambda} : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ be non-constant meromorphic mappings. Let $g_i : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ ($0 \leq i \leq q - 1$) be moving targets located in general position and $T(r, g_i) = o(\max_{1 \leq j \leq \lambda} T(r, f_j))$ ($0 \leq i \leq q - 1$). Assume that $(f_i, g_j) \not\equiv 0$ for $1 \leq i \leq \lambda$, $0 \leq j \leq q - 1$ and $A_j := (f_1, g_j)^{-1}\{0\} = \cdots = (f_{\lambda}, g_j)^{-1}\{0\}$ for each $0 \leq j \leq q - 1$. Denote $\mathcal{A} = \bigcup_{j=0}^{q-1} A_j$. Let l , $2 \leq l \leq \lambda$, be an integer such that for any increasing sequence $1 \leq j_1 < \cdots < j_l \leq \lambda$, $f_{j_1}(z) \wedge \cdots \wedge f_{j_l}(z) = 0$ for every point $z \in \mathcal{A}$.*

If $q > \frac{dN(2N+1)\lambda}{\lambda - l + 1}$, then f_1, \dots, f_{λ} are algebraically over \mathbf{C} , i.e. $f_1 \wedge \cdots \wedge f_{\lambda} \equiv 0$.

Furthermore, in the case of $d = 1$, P. D. Thoan, P. V. Duc and S. D. Quang [10] proved an better algebraic dependences theorem as follows.

Theorem B (see [10, Theorem 1]). *Let $f_1, \dots, f_{\lambda} : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ be non-constant meromorphic mappings. Let $g_i : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})^*$ ($0 \leq i \leq q - 1$) be moving hyperplanes in general position such that $T(r, g_i) = o(\max_{1 \leq j \leq \lambda} T(r, f_j))$ ($0 \leq i \leq q - 1$) and $(f_i, g_j) \not\equiv 0$ for $1 \leq i \leq \lambda$, $0 \leq j \leq q - 1$. Assume that the following conditions are satisfied.*

- (a) $\min\{1, \nu_{(f_1, g_j)}\} = \cdots = \min\{1, \nu_{(f_{\lambda}, g_j)}\}$ for each $0 \leq j \leq q - 1$,
- (b) $\dim\{z \mid (f_1, g_i)(z) = (f_1, g_j)(z) = 0\} \leq n - 2$ for each $0 \leq i < j \leq q - 1$,
- (c) *there exists an integer number l , $2 \leq l \leq \lambda$, such that for any increasing sequence $1 \leq j_1 < \cdots < j_l \leq \lambda$, $f_{j_1}(z) \wedge \cdots \wedge f_{j_l}(z) = 0$ for every point $z \in \bigcup_{i=0}^{q-1} (f_1, g_i)^{-1}\{0\}$.*

If $q > \frac{N(2N+1)\lambda - (N-1)(\lambda-1)}{\lambda - l + 1}$, then $f_1 \wedge \cdots \wedge f_{\lambda} \equiv 0$.

The above results are the best results on the algebraic dependences of meromorphic mappings sharing moving hyperplanes available at the present. However, in our opinion, they are still weak. Also in the above results, all intersecting points of the mappings and the moving hyperplanes are considered. Actually, there are many authors consider the multiple values for meromorphic mappings sharing hyperplanes, i.e., consider only the intersecting points of the mappings f_i and the hyperplanes g_j with the multiplicity not exceed a certain number $k_j < +\infty$. For example, in 2010, T. B. Cao and H. X. Yi gave some uniqueness theorems for meromorphic mappings sharing fixed hyperplanes where all intersecting points more than a certain number are omitted (see [1, Theorems 1.4 and 1.5]). Recently, H. H. Giang consider the multiple values and uniqueness problems for the mappings sharing moving hyperplanes (see [4, Theorems 1.1 and 1.3]).

Our first purpose in this paper is to generalize and improve Theorems A and B by considering the multiple values problem and reducing the number of hyperplanes. Namely, we will prove the following.

Theorem 1.1. *Let $f_1, \dots, f_\lambda : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ be non-constant meromorphic mappings. Let $\{g_j\}_{j=0}^{q-1}$ be moving hyperplanes of $\mathbf{P}^N(\mathbf{C})$ in general position satisfying $T(r, g_j) = o(\max_{1 \leq i \leq \lambda} T(r, f_i))$ ($0 \leq j \leq q-1$). Let k_j ($0 \leq j \leq q-1$) be positive integers or $+\infty$. Assume that $(f_i, g_j) \not\equiv 0$ for $1 \leq i \leq \lambda$, $0 \leq j \leq q-1$ and $A_j := \text{Supp } \nu_{(f_1, g_j), \leq k_j}^0 = \dots = \text{Supp } \nu_{(f_\lambda, g_j), \leq k_j}^0$ for each $0 \leq j \leq q-1$. Denote $\mathcal{A} = \bigcup_{j=0}^{q-1} A_j$. Let l , $2 \leq l \leq \lambda$, be an integer such that for any increasing sequence $1 \leq i_1 < \dots < i_l \leq \lambda$, $f_{i_1}(z) \wedge \dots \wedge f_{i_l}(z) = 0$ for every point $z \in \mathcal{A}$. If*

$$\sum_{j=0}^{q-1} \frac{1}{k_j} < \frac{q}{N(N+2)} - \frac{d\lambda}{\lambda-l+1},$$

then $f_1 \wedge \dots \wedge f_\lambda \equiv 0$.

In the above result, letting $k_j = +\infty$ ($0 \leq j \leq q-1$), we get the conclusion of Theorem A with $q > \frac{d\lambda N(N+2)}{(\lambda-l+1)}$. Also, letting $d = 1$, we obtain the conclusion of Theorem B with $q > \frac{\lambda N(N+2)}{\lambda-l+1}$. Hence our result is improvement of Theorems A and B.

Let $\lambda = l = 2$ and $k_j = +\infty$ ($0 \leq j \leq q-1$), the above theorem implies the following unicity theorem.

Corollary 1.2. *Let $f_1, f_2 : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ be two non-constant meromorphic mappings. Let $\{g_j\}_{j=0}^{q-1}$ be moving hyperplanes of $\mathbf{P}^N(\mathbf{C})$ in general position satisfying $T(r, g_j) = o(\max_{1 \leq i \leq 2} T(r, f_i))$ ($0 \leq j \leq q-1$). Assume that the following conditions are satisfied.*

- (a) $\min\{1, \nu_{(f_1, g_j)}^0\} = \min\{1, \nu_{(f_2, g_j)}^0\}$ for each $0 \leq j \leq q-1$,
- (b) $f_1(z) = f_2(z)$ for each $z \in \bigcup_{j=0}^{q-1} \text{Supp } \nu_{(f_1, g_j)}^0$,
- (c) $q > 2dN(N+2)$.

Then $f_1 \equiv f_2$.

If $d = 1$, then from the above corollary, we get a uniqueness theorem for meromorphic mappings, which are not assumed to be nondegenerate, sharing $q > 2N^2 + 4N$ moving hyperplanes in general position.

Now, if we assume further that the linearly closures of the images of the mappings f_i in Theorem 1.1 have the same dimension then we will get a better result as follows.

Theorem 1.3. *Let $f_1, \dots, f_\lambda : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ be non-constant meromorphic mappings. Let $\{g_j\}_{j=0}^{q-1}$ be q meromorphic mappings $g_j : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})^*$ ($0 \leq j \leq q-1$) in general position satisfying $T(r, g_j) = o(\max_{1 \leq i \leq \lambda} T(r, f_i))$ ($0 \leq j \leq q-1$). Let k_j ($0 \leq j \leq q-1$) be positive integers or $+\infty$. Assume that $(f_i, g_j) \not\equiv 0$ for $1 \leq i \leq \lambda$, $0 \leq j \leq q-1$, and the following conditions are satisfied.*

- (a) $\min\{1, \nu_{(f_1, g_j), \leq k_j}^0\} = \dots = \min\{1, \nu_{(f_\lambda, g_j), \leq k_j}^0\}$ for each $0 \leq j \leq q-1$,
- (b) $\dim f_1^{-1}(g_{j_1}) \cap f_1^{-1}(g_{j_2}) \leq n-2$ for each $0 \leq j_1 < j_2 \leq q-1$,
- (c) *there exists an integer number l , $2 \leq l \leq \lambda$, such that for any increasing sequence*

$$1 \leq i_1 < \dots < i_l \leq \lambda, \quad f_{i_1}(z) \wedge \dots \wedge f_{i_l}(z) = 0 \text{ for every point } z \in \bigcup_{j=0}^{q-1} f_1^{-1}(g_j).$$

We assume further that $\text{rank}_{\mathcal{R}\{g_j\}} f_1 = \cdots = \text{rank}_{\mathcal{R}\{g_j\}} f_\lambda = m + 1$, where m is a positive integer. If

$$(*) \quad \sum_{j=0}^{q-1} \frac{1}{k_j + 1 - m} < \frac{q}{m(2N - m + 2)} - \frac{\lambda q}{q(\lambda - l + 1) + \lambda(m - 1)}$$

then $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$.

Remark 1: i) When $k_0 = \cdots = k_{q-1} = +\infty$, the condition $(*)$ becomes

$$q > \frac{\lambda(2Nm - m^2 + m + 1)}{(\lambda - l + 1)}.$$

We see that this inequality is satisfied with $q > \frac{\lambda(N^2 + N + 1)}{(\lambda - l + 1)}$, since its right hand side attains maximum at $m = N$. Hence in this case ($d = 1$ and $k_0 = \cdots = k_{q-1} = +\infty$), the conclusion of Theorem 1.3 is better than that of Theorem 1.1.

ii) Let $\lambda = l = 2$ and $k_0 = \cdots = k_{q-1} = +\infty$. We may show that if f_1 and f_2 satisfy the conditions (i)-(iii) of Theorem 1.3 and $q > N(N + 2)$ then $\text{rank}_{\mathcal{R}\{g_j\}} f_1 = \text{rank}_{\mathcal{R}\{g_j\}} f_2$.

Indeed, suppose that there are $a_0, \dots, a_N \in \mathcal{R}\{g_j\}$, not all zeros, satisfying $\sum_{0 \leq i \leq N} a_i f_{1i} \equiv 0$. Set $P = \sum_{0 \leq i \leq N} a_i f_{2i}$. Then $P = 0$ on $\bigcup_{1 \leq j \leq q-1} (f_1, g_j)^{-1}(0)$. If $P \not\equiv 0$, then by using Remark 2 below, we have

$$\begin{aligned} T_{f_2}(r) &\geq N(r, \nu_P^0) + o(T_{f_2}(r)) \geq \sum_{j=0}^{q-1} N^{[1]}(r, \nu_{(f_1, g_j)}^0) + o(T_{f_2}(r)) \\ &= \sum_{j=0}^{q-1} N^{[1]}(r, \nu_{(f_2, g_j)}^0) + o(T_{f_2}(r)) \geq \frac{q}{N(N + 2)} T_{f_2}(r) + o(T_{f_2}(r)). \end{aligned}$$

Letting $r \rightarrow +\infty$, we get $q \leq N(N + 2)$. This is a contradiction. Then we must have $P \equiv 0$. This implies that $\text{rank}_{\mathcal{R}\{g_j\}} f_1 \geq \text{rank}_{\mathcal{R}\{g_j\}} f_2$.

Similarly, we have $\text{rank}_{\mathcal{R}\{g_j\}} f_1 \leq \text{rank}_{\mathcal{R}\{g_j\}} f_2$. Hence $\text{rank}_{\mathcal{R}\{g_j\}} f_1 = \text{rank}_{\mathcal{R}\{g_j\}} f_2$.

Then from Theorem 1.3 and the above remarks, we get a uniqueness theorem as follows.

Corollary 1.4. *Let $f_1, f_2 : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ be non - constant meromorphic mappings. Let $\{g_j\}_{j=0}^{q-1}$ be moving hyperplanes of $\mathbf{P}^N(\mathbf{C})$ in general position satisfying $T(r, g_j) = o(\max_{1 \leq i \leq 2} T(r, f_i))$ ($0 \leq j \leq q - 1$). Assume that $(f_i, g_j) \not\equiv 0$ for $1 \leq i \leq 2$, $0 \leq j \leq q - 1$, and the following conditions are satisfied.*

- (a) $\min\{1, \nu_{(f_1, g_j)}^0\} = \min\{1, \nu_{(f_2, g_j)}^0\}$ for each $0 \leq j \leq q - 1$,
- (b) $\dim f_1^{-1}(g_{j_1}) \cap f_1^{-1}(g_{j_2}) \leq n - 2$ for each $0 \leq j_1 < j_2 \leq q - 1$,
- (c) $f_1 = f_2$ on $\bigcup_{j=0}^{q-1} \text{Supp } \nu_{(f_1, g_j)}^0$.

If $q > 2N^2 + 2N + 2$, then $f_1 \equiv f_2$.

We would like to note that there are several results on the uniqueness problem of meromorphic mappings sharing moving hyperplanes regardless of multiplicity. For example, in 2007, with the same assumption of Corollary 1.4 (and plus $N \geq 2$), Z. Chen, Y. Li and

Q. Yan [2] get the uniqueness theorem with $q \geq 4N^2 + 2N$. In 2013, P. D. Thoan, P. V. Duc and S. D. Quang improved the result of these authors to the case of $q \geq 4N^2 + 2$ (for any N). Therefore, our above uniqueness theorem is much stronger improvements of many previous results.

Acknowledgements: This paper was supported in part by a NAFOSTED grant of Vietnam.

2. BASIC NOTIONS AND AUXILIARY RESULTS FROM NEVANLINNA THEORY

2.1. We set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$ for $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ and define

$$B(r) := \{z \in \mathbf{C}^n : \|z\| < r\}, \quad S(r) := \{z \in \mathbf{C}^n : \|z\| = r\} \quad (0 < r < \infty).$$

Define

$$v_{n-1}(z) := (dd^c \|z\|^2)^{n-1} \quad \text{and} \\ \sigma_n(z) := d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1} \text{ on } \mathbf{C}^n \setminus \{0\}.$$

2.2. For a divisor ν on \mathbf{C}^n , we denote by $N(r, \nu)$ the counting function of the divisor ν as usual in Nevanlinna theory (see [5]).

For a positive integer M or $M = \infty$, we define the truncated divisors of ν by

$$\nu^{[M]}(z) = \min \{M, \nu(z)\}, \quad \nu_{\leq k}^{[M]}(z) := \begin{cases} \nu^{[M]}(z) & \text{if } \nu^{[M]}(z) \leq k, \\ 0 & \text{if } \nu^{[M]}(z) > k. \end{cases}$$

Similarly, we define $\nu_{>k}^{[M]}$. We will write $N^{[M]}(r, \nu)$, $N_{\leq k}^{[M]}(r, \nu)$, $N_{>k}^{[M]}(r, \nu)$ for $N(r, \nu^{[M]})$, $N(r, \nu_{\leq k}^{[M]})$, $N(r, \nu_{>k}^{[M]})$ as respectively. We will omit the character $^{[M]}$ if $M = \infty$.

2.3. Let $f : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_N)$ on $\mathbf{P}^N(\mathbf{C})$, we take a reduced representation $f = (f_0 : \cdots : f_N)$, which means that each f_i is a holomorphic function on \mathbf{C}^n and $f(z) = (f_0(z) : \cdots : f_N(z))$ outside the analytic set $\{f_0 = \cdots = f_N = 0\}$ of codimension ≥ 2 . Let a be a meromorphic mapping of \mathbf{C}^n into $\mathbf{P}^N(\mathbf{C})^*$ with reduced representation $a = (a_0 : \cdots : a_N)$. We denote by $T(r, f)$ the characteristic function of f and by $m_{f,a}(r)$ the proximity function of f with respect to a (see [9]).

If $(f, a) \not\equiv 0$, then the first main theorem for moving targets in value distribution theory states

$$T(r, f) + T(r, a) = m_{f,a}(r) + N_{(f,a)}(r).$$

Theorem 2.1 (The First Main Theorem for general position [7, p. 326]). *Let $f_j : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$, $1 \leq j \leq k$ be meromorphic mappings located in general position. Assume that $1 \leq k \leq N$. Then*

$$N(r, \mu_{f_1 \wedge \cdots \wedge f_\lambda}) + m(r, f_1 \wedge \cdots \wedge f_\lambda) \leq \sum_{1 \leq i \leq \lambda} T(r, f_i) + O(1).$$

Here, by $\mu_{f_1 \wedge \cdots \wedge f_\lambda}$ we denote the divisor associated with $f_1 \wedge \cdots \wedge f_\lambda$. We also denote by $N_{f_1 \wedge \cdots \wedge f_\lambda}(r)$ the counting function associated with the divisor $\mu_{f_1 \wedge \cdots \wedge f_\lambda}$.

Let V be a complex vector space of dimension $N \geq 1$. The vectors $\{v_1, \dots, v_k\}$ are said to be in general position if for each selection of integers $1 \leq i_1 < \cdots < i_p \leq k$ with $p \leq N$,

then $v_{i_1} \wedge \cdots \wedge v_{i_p} \neq 0$. The vectors $\{v_1, \dots, v_k\}$ are said to be in special position if they are not in general position. Take $1 \leq p \leq k$. Then $\{v_1, \dots, v_k\}$ are said to be in p -special position if for each selection of integers $1 \leq i_1 < \cdots < i_p \leq k$, the vectors v_{i_1}, \dots, v_{i_p} are in special position.

Theorem 2.2 (The Second Main Theorem for general position [7, p. 320]). *Let M be a connected complex manifold of dimension m . Let A be a pure $(m-1)$ -dimensional analytic subset of M . Let V be a complex vector space of dimension $n+1 > 1$. Let p and k be integers with $1 \leq p \leq k \leq n+1$. Let $f_j : M \rightarrow P(V), 1 \leq j \leq k$, be meromorphic mappings. Assume that f_1, \dots, f_k are in general position. Also assume that f_1, \dots, f_k are in p -special position on A . Then we have*

$$\mu_{f_1 \wedge \cdots \wedge f_k} \geq (k-p+1)\nu_A.$$

Here by ν_A we denote the reduced divisor whose support is the set A .

The following is a new second main theorem given by S. D. Quang [9], which is an improvement the second main theorem of Thai-Quang in [11].

Theorem 2.3 (The Second Main Theorem for moving target [9]). *Let $f : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^q$ ($q \geq 2N-m+2$) be meromorphic mappings of \mathbf{C}^n into $\mathbf{P}^N(\mathbf{C})^*$ in general position such that $(f, a_i) \not\equiv 0$ ($1 \leq i \leq q$), where $m+1 = \text{rank}_{\mathcal{R}\{a_i\}}(f)$. Then we have*

$$\| \frac{q}{2N-m+2} T_f(r) \leq \sum_{i=1}^q N_{(f, a_i)}^{[m]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)).$$

As usual, by the notation “ $\| P$ ” we mean the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

Remark 2: With the assumption of Theorem 2.3, we see that

$$\begin{aligned} \| T_f(r) &\leq \frac{m(2N-m+2)}{q} \sum_{i=1}^q N_{(f, a_i)}^{[1]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)) \\ (2.4) \quad &\leq \frac{N(N+2)}{q} \sum_{i=1}^q N_{(f, a_i)}^{[1]}(r) + o(T_f(r)) + O(\max_{1 \leq i \leq q} T_{a_i}(r)). \end{aligned}$$

3. PROOFS OF MAIN THEOREMS

3.1. Proof of Theorem 1.1.

It suffices to prove Theorem 1.1 in the case of $\lambda \leq N+1$.

Suppose that $f_1 \wedge \cdots \wedge f_\lambda \not\equiv 0$. We now prove the following.

Claim 3.1. *For every $1 \leq i \leq \lambda$, $0 \leq j \leq q-1$ and $1 \leq m \leq N$, we have*

$$N_{\leq k_j}^{[m]}(r, \nu_{(f_i, g_j)}^0) \geq \frac{k_j+1}{k_j+1-m} N^{[m]}(r, \nu_{(f_i, g_j)}^0) - \frac{m}{k_j+1-m} T(r, f_i).$$

Indeed, we have

$$\begin{aligned}
N_{\leq k_j}^{[m]}(r, \nu_{(f_i, g_j)}^0) &= N^{[m]}(r, \nu_{(f_i, g_j)}^0) - N_{> k_j}^{[m]}(r, \nu_{(f_i, g_j)}^0) \\
&\geq N^{[m]}(r, \nu_{(f_i, g_j)}^0) - \frac{m}{k_j + 1} N_{> k_j}(r, \nu_{(f_i, g_j)}^0) \\
&= N^{[m]}(r, \nu_{(f_i, g_j)}^0) - \frac{m}{k_j + 1} N(r, \nu_{(f_i, g_j)}^0) + \frac{m}{k_j + 1} N_{\leq k_j}(r, \nu_{(f_i, g_j)}^0) \\
&\geq N^{[m]}(r, \nu_{(f_i, g_j)}^0) - \frac{m}{k_j + 1} T(r, f_i) + \frac{m}{k_j + 1} N_{\leq k_j}^{[N]}(r, \nu_{(f_i, g_j)}^0).
\end{aligned}$$

Thus

$$N_{\leq k_j}^{[m]}(r, \nu_{(f_i, g_j)}^0) \geq \frac{k_j + 1}{k_j + 1 - m} N^{[m]}(r, \nu_{(f_i, g_j)}^0) - \frac{m}{k_j + 1 - m} T(r, f_i).$$

Then the claim is proved.

Claim 3.2. *For every $1 \leq i \leq \lambda$, we have*

$$\sum_{j=0}^{q-1} \min\{1, \nu_{(f_i, g_j)}^0\}_{\leq k_j} \leq \frac{d}{\lambda - l + 1} \mu_{f_1 \wedge \dots \wedge f_\lambda}(z) + q \cdot \sum_{\beta} \mu_{g_{\beta(1)} \wedge \dots \wedge g_{\beta(N+1)}}(z)$$

for each $z \notin A \cup_{i=1}^{\lambda} I(f_i)$, where the sum is over all injective maps $\beta : \{1, 2, \dots, N+1\} \rightarrow \{1, 2, \dots, q\}$

Indeed, for each regular point $z_0 \in \mathcal{A} \setminus (A \cup \bigcup_{i=1}^{\lambda} I(f_i) \cup \bigcup_{\beta \in T[N+1, q]} \{z | g_{\beta(1)}(z) \wedge \dots \wedge g_{\beta(N+1)}(z) = 0\})$ and for each increasing sequence $1 \leq i_1 < \dots < i_l \leq \lambda$, we have

$$f_{i_1}(z_0) \wedge \dots \wedge f_{i_l}(z_0) = 0.$$

By the Second Main Theorem for general position [7, p. 320], we have

$$\mu_{f_1 \wedge \dots \wedge f_\lambda}(z_0) \geq \lambda - (l - 1).$$

Hence

$$\sum_{j=0}^{q-1} \min\{1, \nu_{(f_i, g_j)}^0\}_{\leq k_j}(z_0) \leq \sum_{j=0}^{q-1} \min\{1, \nu_{(f_i, g_j)}^0(z_0)\} \leq d \leq \frac{d}{\lambda - l + 1} \mu_{f_1 \wedge \dots \wedge f_\lambda}(z_0).$$

If $z_0 \in \bigcup_{\beta \in T[N+1, q]} \{z | g_{\beta(1)}(z) \wedge \dots \wedge g_{\beta(N+1)}(z) = 0\}$, then we have

$$\sum_{j=0}^{q-1} \min\{1, \nu_{(f_i, g_j)}^0\}_{\leq k_j}(z_0) \leq \sum_{j=0}^{q-1} \min\{1, \nu_{(f_i, g_j)}^0(z_0)\} \leq q \sum_{\beta \in T[N+1, q]} \mu_{g_{\beta(1)} \wedge \dots \wedge g_{\beta(N+1)}}(z_0).$$

Thus, for each $z \notin A \cup \bigcup_{i=1}^{\lambda} I(f_i)$, we have

$$\sum_{j=0}^{q-1} \min\{1, \nu_{(f_i, g_j)}^0\}_{\leq k_j}(z) \leq \frac{d}{\lambda - l + 1} \mu_{f_1 \wedge \dots \wedge f_\lambda}(z) + q \sum_{\beta \in T[N+1, q]} \mu_{g_{\beta(1)} \wedge \dots \wedge g_{\beta(N+1)}}(z).$$

The Claim 3.2 is proved.

The above Claim yields that

$$\begin{aligned}
\| \sum_{j=0}^{q-1} N_{\leq k_j}^{[1]}(r, \nu_{(f_i, g_j)}^0) &\leq \frac{d}{\lambda - l + 1} N_{f_1 \wedge \dots \wedge f_\lambda}(r) + q \sum_{\beta \in T[N+1, q]} N_{g_{\beta(1)} \wedge \dots \wedge g_{\beta(N+1)}}(r) \\
&\leq \frac{d}{\lambda - l + 1} \sum_{i=1}^{\lambda} T(r, f_i) + q \sum_{\beta \in T[N+1, q]} \sum_{i=1}^{N+1} T(r, g_{\beta(i)}) \\
&= \frac{d}{\lambda - l + 1} T(r) + o(\max_{1 \leq i \leq \lambda} T(r, f_i)).
\end{aligned}$$

where $T(r) = \sum_{i=1}^{\lambda} T(r, f_i)$. Then, by Claim 3.1 and the inequality (2.4) we have

$$\begin{aligned}
\| \frac{d\lambda}{\lambda - l + 1} T(r) &\geq \sum_{i=1}^{\lambda} \sum_{j=0}^{q-1} N_{\leq k_j}^{[1]}(r, \nu_{(f_i, g_j)}^0) + o(T(r)) \\
&\geq \sum_{i=1}^{\lambda} \sum_{j=0}^{q-1} \left(\frac{k_j + 1}{k_j} N_{\leq k_j}^{[1]}(r, \nu_{(f_i, g_j)}^0) - \frac{1}{k_j} T(r, f_i) \right) + o(T(r)) \\
&\geq \sum_{i=1}^{\lambda} \sum_{j=0}^{q-1} N_{\leq k_j}^{[1]}(r, \nu_{(f_i, g_j)}^0) - \sum_{j=0}^{q-1} \frac{1}{k_j} T(r) + o(T(r)) \\
&\geq \sum_{i=1}^{\lambda} \frac{q}{N(N+2)} T(r, f_i) - \sum_{j=0}^{q-1} \frac{1}{k_j} T(r) + o(T(r)) \\
&= \left(\frac{q}{N(N+2)} - \sum_{j=0}^{q-1} \frac{1}{k_j} \right) T(r) + o(T(r)).
\end{aligned}$$

Letting $r \rightarrow +\infty$, we get

$$\sum_{j=0}^{q-1} \frac{1}{k_j} \geq \frac{q}{N(N+2)} - \frac{d\lambda}{\lambda - l + 1}.$$

This is a contradiction. Thus, $f_1 \wedge \dots \wedge f_\lambda = 0$. The theorem is proved. \square

3.2. Proof of Theorem 1.3.

It suffices to prove Theorem 1.3 in the case of $\lambda \leq N+1$. We set $m = \text{rank}_{\mathcal{R}_{\{g_j\}}}(f_i) - 1$. Similar as [3, Claim 3.1], we have the following claim.

Claim 3.3. *Let $h_i : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})$ ($1 \leq i \leq p \leq N+1$) be meromorphic mappings with reduced representations $h_i := (h_{i0} : \dots : h_{iN})$. Let $a_i : \mathbf{C}^n \rightarrow \mathbf{P}^N(\mathbf{C})^*$ ($1 \leq i \leq N+1$) be moving hyperplanes with reduced representations $a_i := (a_{i0} : \dots : a_{iN})$. Put $\tilde{h}_i := ((h_i, a_1) : \dots : (h_i, a_{N+1}))$. Assume that a_1, \dots, a_{N+1} are located in general position such that $(h_i, a_j) \not\equiv 0$ ($1 \leq i \leq p$, $1 \leq j \leq N+1$). Let S be a pure $(n-1)$ -dimensional analytic subset of \mathbf{C}^n such that $S \not\subset (a_1 \wedge \dots \wedge a_{N+1})^{-1}\{0\}$. Then $h_1 \wedge \dots \wedge h_p = 0$ on S if and only if $\tilde{h}_1 \wedge \dots \wedge \tilde{h}_p = 0$ on S .*

We now prove Claim 3.3 .

(\Rightarrow) Suppose that $\{\tilde{h}_1 \wedge \cdots \wedge \tilde{h}_p\} \not\equiv 0$ on S . Then there exists $z_0 \in S$ such that $\tilde{h}_1(z_0) \wedge \cdots \wedge \tilde{h}_p(z_0) \neq 0$. This means that the family $\{\tilde{h}_1(z_0), \dots, \tilde{h}_p(z_0)\}$ is linearly independent on \mathbf{C} , i.e., the following matrix is of rank p

$$\begin{aligned} A &= \begin{pmatrix} (h_1, a_1)(z_0) & \cdots & (h_p, a_1)(z_0) \\ \vdots & \vdots & \vdots \\ (h_1, a_{N+1})(z_0) & \cdots & (h_p, a_{N+1})(z_0) \end{pmatrix} \\ &= \begin{pmatrix} a_{10}(z_0) & \cdots & a_{1N}(z_0) \\ \vdots & \vdots & \vdots \\ a_{N+10}(z_0) & \cdots & a_{N+1N}(z_0) \end{pmatrix} \cdot \begin{pmatrix} h_{10}(z_0) & \cdots & h_{p0}(z_0) \\ \vdots & \vdots & \vdots \\ h_{1N}(z_0) & \cdots & h_{pN}(z_0) \end{pmatrix}. \end{aligned}$$

Hence the matrix

$$\begin{pmatrix} h_{10}(z_0) & \cdots & h_{p0}(z_0) \\ \vdots & \vdots & \vdots \\ h_{1N}(z_0) & \cdots & h_{pN}(z_0) \end{pmatrix}$$

is of rank p , i.e., $h_1(z_0) \wedge \cdots \wedge h_p(z_0) \neq 0$. This yields that $h_1 \wedge \cdots \wedge h_p \not\equiv 0$ on S . This is a contradiction.

(\Leftarrow) We see that the following matrix is of rank $\leq p-1$ for each $z \in S$

$$A = \begin{pmatrix} (h_1, a_1)(z) & \cdots & (h_p, a_1)(z) \\ \vdots & \vdots & \vdots \\ (h_1, a_{N+1})(z) & \cdots & (h_p, a_{N+1})(z) \end{pmatrix}.$$

On the other hand, we have

$$A = \begin{pmatrix} a_{10}(z) & \cdots & a_{1N}(z) \\ \vdots & \vdots & \vdots \\ a_{N+10}(z) & \cdots & a_{N+1N}(z) \end{pmatrix} \cdot \begin{pmatrix} h_{10}(z) & \cdots & h_{p0}(z) \\ \vdots & \vdots & \vdots \\ h_{1N}(z) & \cdots & h_{pN}(z) \end{pmatrix}.$$

Since the family $\{a_i\}$ is located in general position and $S \not\subset (a_1 \wedge \cdots \wedge a_{N+1})^{-1}\{0\}$, this implies that the matrix

$$\begin{pmatrix} h_{10}(z) & \cdots & h_{p0}(z) \\ \vdots & \vdots & \vdots \\ h_{1N}(z) & \cdots & h_{pN}(z) \end{pmatrix}$$

is of rank $\leq p-1$ for each $z \in S$, i.e., $h_1 \wedge \cdots \wedge h_p \equiv 0$ on S .

The Claim 3.3 is proved.

We now continue to prove the theorem. Suppose that $f_1 \wedge \cdots \wedge f_\lambda \not\equiv 0$.

For λ indices $0 = j_0 < j_1 < \cdots < j_{\lambda-1} \leq N$ such that

$$\begin{pmatrix} (f_1, g_{j_0}) & \cdots & (f_\lambda, g_{j_0}) \\ (f_1, g_{j_1}) & \cdots & (f_\lambda, g_{j_1}) \\ \vdots & \vdots & \vdots \\ (f_1, g_{j_{\lambda-1}}) & \cdots & (f_\lambda, g_{j_{\lambda-1}}) \end{pmatrix}$$

is nondegenerate.

$$\text{Put } J = \{j_0, \dots, j_{\lambda-1}\}, J^c = \{0, \dots, q-1\} \setminus J \text{ and } B_J = \begin{pmatrix} (f_1, g_{j_0}) & \cdots & (f_\lambda, g_{j_0}) \\ (f_1, g_{j_1}) & \cdots & (f_\lambda, g_{j_1}) \\ \vdots & \vdots & \vdots \\ (f_1, g_{j_{\lambda-1}}) & \cdots & (f_\lambda, g_{j_{\lambda-1}}) \end{pmatrix}.$$

We now prove the following claim.

Claim 3.4. *If B_J is nondegenerate, i.e., $\det B_J \neq 0$ then*

$$\begin{aligned} \sum_{j \in J} (\min_{1 \leq i \leq \lambda} \{\nu_{(f_i, g_j), \leq k_j}^0\} - \min\{1, \nu_{(f_1, g_j), \leq k_j}^0\}) \\ + \sum_{j=0}^{q-1} (\lambda - l + 1) \min\{1, \nu_{(f_1, g_j), \leq k_j}^0\} \leq \mu_{\tilde{f}_1 \wedge \dots \wedge \tilde{f}_\lambda} \end{aligned}$$

on the set $\mathbf{C}^n \setminus (A \cup \bigcup_{i=1}^\lambda I(f_i) \cup (g_{j_0} \wedge \dots \wedge g_{j_{\lambda-1}})^{-1}(0))$, where $\tilde{f}_i := ((f_i, g_{j_0}) : \dots : (f_i, g_{j_{\lambda-1}}))$ and $A = \bigcup_{0 \leq i < j \leq q-1} Z_{(f_1, g_i)} \cap Z_{(f_1, g_j)}$.

Indeed, put $\mathcal{A} := \bigcup_{j \in J} Z_{(f_1, g_j)}$, $\mathcal{A}^c := \bigcup_{j \in J^c} Z_{(f_1, g_j)}$. We consider the following two cases.

Case 1. Let $z_0 \in \mathcal{A} \setminus (A \cup \bigcup_{i=1}^\lambda I(f_i) \cup (g_{j_0} \wedge \dots \wedge g_{j_{\lambda-1}})^{-1}(0))$ be a regular point of \mathcal{A} . Then z_0 is a zero of one of the meromorphic functions $\{(f_1, g_j)\}_{j \in J}$. Without loss of generality we may assume that z_0 is a zero of (f_1, g_{j_0}) . Let S be an irreducible component of \mathcal{A} containing z_0 . Suppose that U is an open neighborhood of z_0 in \mathbf{C}^n such that $U \cap \{\mathcal{A} \setminus S\} = \emptyset$. Choose a holomorphic function h on an open neighborhood $U' \subset U$ of z_0 such that $\nu_h(z) = \min_{1 \leq i \leq \lambda} \{\nu_{(f_i, g_{j_0}), \leq k_{j_0}}^0(z)\}$ if $z \in S$ and $\nu_h(z) = 0$ if $z \notin S$. Then $(f_i, g_{j_0}) = a_i h$ ($1 \leq i \leq \lambda$), where a_i are holomorphic functions. Therefore, the matrix $\begin{pmatrix} (f_1, g_{j_1}) & \cdots & (f_\lambda, g_{j_1}) \\ \vdots & \vdots & \vdots \\ (f_1, g_{j_{\lambda-1}}) & \cdots & (f_\lambda, g_{j_{\lambda-1}}) \end{pmatrix}$ is of rank $\leq \lambda - 1$. Hence, there exist λ holomorphic functions b_1, \dots, b_λ , not all zeros, such that

$$\sum_{i=1}^\lambda b_i \cdot (f_i, g_{j_k}) = 0 \quad (1 \leq k \leq \lambda - 1).$$

Without loss of generality, we may assume that the set of common zeros of $\{b_i\}_{i=1}^\lambda$ is an analytic subset of codimension ≥ 2 . Then there exists an index $i_1, 1 \leq i_1 \leq \lambda$, such that $S \not\subset b_{i_1}^{-1}\{0\}$. We may assume that $i_1 = \lambda$. Then for each $z \in (U' \cup S) \setminus b_\lambda^{-1}\{0\}$, we have

$$\begin{aligned} \tilde{f}_1(z) \wedge \dots \wedge \tilde{f}_\lambda(z) &= \tilde{f}_1(z) \wedge \dots \wedge \tilde{f}_{\lambda-1}(z) \wedge \left(\tilde{f}_\lambda(z) + \sum_{i=1}^{\lambda-1} \frac{b_i}{b_\lambda} \tilde{f}_i(z) \right) \\ &= \tilde{f}_1(z) \wedge \dots \wedge \tilde{f}_{\lambda-1}(z) \wedge (V(z)h(z)) \\ &= h(z) \cdot (\tilde{f}_1(z) \wedge \dots \wedge \tilde{f}_{\lambda-1}(z) \wedge V(z)), \end{aligned}$$

where $V(z) := (a_\lambda + \sum_{i=1}^{\lambda-1} \frac{b_i}{b_\lambda} a_i, 0, \dots, 0)$.

By the assumption and by Claim 3.3, for any increasing sequence $1 \leq i_1 < \dots < i_l \leq \lambda - 1$, we have $\tilde{f}_{i_1} \wedge \dots \wedge \tilde{f}_{i_l} = 0$ on S . This implies that the family $\{\tilde{f}_1, \dots, \tilde{f}_{\lambda-1}, V\}$ is in

$(l+1)$ -special position on S . By using The Second Main Theorem for general position [7, p. 320], we have

$$\mu_{\tilde{f}_1 \wedge \dots \wedge \tilde{f}_{\lambda-1} \wedge V}(z) \geq \lambda - l, \forall z \in S.$$

Hence

$$\mu_{\tilde{f}_1 \wedge \dots \wedge \tilde{f}_\lambda}(z) \geq \nu_h(z) + \lambda - l = \min_{1 \leq i \leq \lambda} \{\nu_{(f_i, g_{j_0}), \leq k_{j_0}}^0(z)\} + \lambda - l,$$

for all $z \in (U' \cup S) \setminus b_{i_1}^{-1}\{0\}$. This implies that

$$\begin{aligned} \sum_{j \in J} (\min_{1 \leq i \leq \lambda} \{\nu_{(f_i, g_j), \leq k_j}^0(z_0)\} - \min\{1, \nu_{(f_1, g_j), \leq k_j}^0(z_0)\}) \\ + \sum_{j=0}^{q-1} (\lambda - l + 1) \min\{1, \nu_{(f_1, g_j), \leq k_j}^0(z_0)\} \\ = \min_{1 \leq i \leq \lambda} \{\nu_{(f_i, g_{j_0}), \leq k_{j_0}}^0(z_0)\} + \lambda - l \leq \mu_{\tilde{f}_1 \wedge \dots \wedge \tilde{f}_\lambda}(z_0). \end{aligned}$$

Case 2. Let $z_0 \in \mathcal{A}^c \setminus (A \cup \bigcup_{i=1}^\lambda I(f_i) \cup (g_{j_0} \wedge \dots \wedge g_{j_{\lambda-1}})^{-1}(0))$ be a regular point of \mathcal{A}^c . Then z_0 is a zero of $(f_1, g_j), j \in J^c$. By the assumption and by Claim 3.3, the family $\{\tilde{f}_1, \dots, \tilde{f}_\lambda\}$ is in l -special position on each irreducible component of \mathcal{A}^c containing z_0 . By using The Second Main Theorem for general position [7, p. 320], we have

$$\mu_{\tilde{f}_1 \wedge \dots \wedge \tilde{f}_\lambda}(z_0) \geq \lambda - l + 1.$$

Hence

$$\begin{aligned} \sum_{j \in J} (\min_{1 \leq i \leq \lambda} \{\nu_{(f_i, g_j), \leq k_j}^0(z_0)\} - \min\{1, \nu_{(f_1, g_j), \leq k_j}^0(z_0)\}) \\ + \sum_{j=0}^{q-1} (\lambda - l + 1) \min\{1, \nu_{(f_1, g_j), \leq k_j}^0(z_0)\} \\ = \lambda - l + 1 \leq \mu_{\tilde{f}_1 \wedge \dots \wedge \tilde{f}_\lambda}(z_0). \end{aligned}$$

From the above two cases we get the desired inequality of the claim.

We now continue to prove the theorem. For each $j, 0 \leq j \leq q-1$, we set

$$N_j(r) = \sum_{i=1}^\lambda N_{\leq k_j}^{[m]}(r, \nu_{(f_i, g_j)}^0) - ((\lambda - 1)m + 1) N_{\leq k_j}^{[1]}(r, \nu_{(f_1, g_j)}^0).$$

For each permutation $I = (j_0, \dots, j_{q-1})$ of $(0, \dots, q-1)$, we set

$$T_I = \{r \in [1, +\infty); N_{j_0}(r) \geq \dots \geq N_{j_{q-1}}(r)\}.$$

It is clear that $\bigcup_I T_I = [1, +\infty)$. Therefore, there exists a permutation, for instance it is $I_0 = (0, \dots, q-1)$, such that $\int_{T_{I_0}} dr = +\infty$. Then we have

$$N_0(r) \geq N_1(r) \geq \dots \geq N_{q-1}(r) \text{ for all } r \in T_{I_0}.$$

By the assumption for $f_1 \wedge \dots \wedge f_\lambda \not\equiv 0$, there exist indices $J = \{j_0, \dots, j_{\lambda-1}\}$ with $0 = j_0 < j_1 < \dots < j_{\lambda-1} \leq N$ such that $\det B_J \not\equiv 0$. We note that

$$N_0(r) = N_{j_0}(r) \geq N_{j_1}(r) \geq \dots \geq N_{j_{\lambda-1}}(r) \geq N_N(r), \text{ for each } r \in T_{I_0}.$$

We see that $\min_{1 \leq i \leq \lambda} a_i \geq \sum_{i=1}^{\lambda} \min\{m, a_i\} - (\lambda - 1)m$ for every λ non-negative integers a_1, \dots, a_{λ} . Then Claim 3.4 implies that

$$\begin{aligned} \sum_{j \in J} \left(\sum_{i=1}^{\lambda} \min\{m, \nu_{(f_i, g_j), \leq k_j}^0\} - ((\lambda - 1)m + 1) \min\{1, \nu_{(f_1, g_j), \leq k_j}^0\} \right) \\ + \sum_{j=0}^{q-1} (\lambda - l + 1) \min\{1, \nu_{(f_1, g_j), \leq k_j}^0\} \leq \mu_{\tilde{f}_1 \wedge \dots \wedge \tilde{f}_{\lambda}}. \end{aligned}$$

on the set $\mathbf{C}^n \setminus (A \cup \bigcup_{i=1}^{\lambda} I(f_i) \cup (g_{j_0} \wedge \dots \wedge g_{j_{\lambda-1}})^{-1}(0))$. Integrating both sides of this inequality, we have

$$\begin{aligned} \sum_{j \in J} \left(\sum_{i=1}^{\lambda} N_{\leq k_j}^{[m]}(r, \nu_{(f_i, g_j)}^0) - ((\lambda - 1)m + 1) N_{\leq k_j}^{[1]}(r, \nu_{(f_1, g_j)}^0) \right) \\ (3.5) \quad + \sum_{j=0}^{q-1} (\lambda - l + 1) N_{\leq k_j}^{[1]}(r, \nu_{(f_1, g_j)}^0) \leq N_{\tilde{f}_1 \wedge \dots \wedge \tilde{f}_{\lambda}}(r) = N_{\det B_J}(r). \end{aligned}$$

Also, by Jensen's formula, we have

$$(3.6) \quad N_{\det B_J}(r) \leq \int_{S(r)} \log |\det B_J| \sigma_n + O(1) \leq \sum_{i=1}^{\lambda} T(r, f_i) + o(\max_{1 \leq i \leq \lambda} T(r, f_i)).$$

Set $T(r) = \sum_{i=1}^{\lambda} T(r, f_i)$. Combining (3.5) and (3.6), then for all $r \in I_0$, we have

$$\begin{aligned} || \quad T(r) &\geq \sum_{i=0}^{\lambda-1} N_{j_i}(r) + \sum_{j=0}^{q-1} (\lambda - l + 1) N_{\leq k_j}^{[1]}(r, \nu_{(f_1, g_j)}^0) + o(T(r)) \\ &\geq \frac{\lambda}{q} \sum_{j=0}^{q-1} N_j(r) + \sum_{j=0}^{q-1} (\lambda - l + 1) N_{\leq k_j}^{[1]}(r, \nu_{(f_1, g_j)}^0) + o(T(r)) \\ &= \sum_{j=0}^{q-1} \left(\lambda - l + 1 - \frac{\lambda((\lambda - 1)m + 1)}{q} \right) N_{\leq k_j}^{[1]}(r, \nu_{(f_1, g_j)}^0) \\ &\quad + \sum_{j=0}^{q-1} \frac{\lambda}{q} \sum_{i=1}^{\lambda} N_{\leq k_j}^{[m]}(r, \nu_{(f_i, g_j)}^0) + o(T(r)) \\ &\geq \sum_{i=1}^{\lambda} \sum_{j=0}^{q-1} \left(\frac{\lambda}{q} + \frac{\lambda - l + 1}{\lambda m} - \frac{(\lambda - 1)m + 1}{mq} \right) N_{\leq k_j}^{[m]}(r, \nu_{(f_i, g_j)}^0) + o(T(r)) \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i=1}^{\lambda} \sum_{j=0}^{q-1} \frac{q(\lambda-l+1) + \lambda(m-1)}{\lambda m q} (N^{[m]}(r, \nu_{(f_i, g_j)}^0) - \frac{m}{k_j + 1 - m} T(r, f_i)) + o(T(r)) \\
&\geq \frac{q(\lambda-l+1) + \lambda(m-1)}{\lambda m q} \left(\sum_{i=1}^{\lambda} \sum_{j=0}^{q-1} N^{[m]}(r, \nu_{(f_i, g_j)}^0) - \sum_{j=0}^{q-1} \frac{m}{k_j + 1 - m} T(r) \right) + o(T(r)) \\
&\geq \frac{q(\lambda-l+1) + \lambda(m-1)}{\lambda m q} \left(\frac{q}{2N - m + 2} - \sum_{j=0}^{q-1} \frac{m}{k_j + 1 - m} \right) T(r) + o(T(r)).
\end{aligned}$$

Letting $r \rightarrow +\infty$, we have

$$1 \geq \frac{q(\lambda-l+1) + \lambda(m-1)}{\lambda m q} \left(\frac{q}{2N - m + 2} - \sum_{j=0}^{q-1} \frac{m}{k_j + 1 - m} \right).$$

Thus

$$\sum_{j=0}^{q-1} \frac{1}{k_j + 1 - m} \geq \frac{q}{m(2N - m + 2)} - \frac{\lambda q}{q(\lambda-l+1) + \lambda(m-1)}.$$

This is a contradiction. Thus, we have $f_1 \wedge \cdots \wedge f_{\lambda} \equiv 0$. \square

REFERENCES

- [1] T. B. Cao and H. X. Yi, *Uniqueness theorems for meromorphic mappings sharing hyperplanes in general position*, arXiv:1011.5828v4 [math.CV].
- [2] Z. Chen and Y. Li and Q. Yan, *Uniqueness problem with truncated multiplicities of meromorphic mappings for moving targets*, Acta Math. Sci. Ser. B Engl. Ed. **27** (2007), 625-634.
- [3] P. V. Duc and P. D. Thoan, *Algebraic dependences of meromorphic mappings in several complex variables*, Ukrain. Math. J. **62** (2010), 92-936.
- [4] H. H. Giang, *Multiple values and finiteness problem of meromorphic mappings sharing different families of moving hyperplanes*, arXiv:1404.0177v1 [math.CV], to appear in Bull. Math. Soc. Sci. Math. Roumanie.
- [5] J. Noguchi and T. Ochiai, *Introduction to Geometric Function Theory in Several Complex Variables*, Trans. Math. Monogr. 80, Amer. Math. Soc., Providence, Rhode Island, 1990.
- [6] M. Ru, *A uniqueness theorem with moving targets without counting multiplicity*, Proc. Amer. Math. Soc. **129** (2001), 2701-2707.
- [7] W. Stoll, *On the propagation of dependences*, Pacific J. Math., **139** (1989), 311-337.
- [8] S. D. Quang, *Algebraic dependences of meromorphic mappings sharing few moving hyperplanes*, Ann. Polon. Math. **108** (2013), 61-73.
- [9] S. D. Quang, *Second main theorem for meromorphic mappings intersecting moving hyperplanes with truncated counting functions and unicity problem*, Preprint, arXiv:1402.3156 [math. CV].
- [10] P. D. Thoan and P. V. Duc and S. D. Quang, *Algebraic dependence and unicity problem with a truncation level to 1 of meromorphic mappings sharing moving targets*, Bull. Math. Soc. Sci. Math. Roumanie **56**(104) (2013), 513-526
- [11] D. D. Thai and S. D. Quang, *Second main theorem with truncated counting function in several complex variables for moving targets*, Forum Math. **20** (2008) 163-179.

FACULTY OF EDUCATION, AN GIANG UNIVERSITY, 18 UNG VAN KHIEM, DONG XUYEN, LONG XUYEN, AN GIANG, VIETNAM

E-mail address: nquynh1511@gmail.com